ENGINEERING MATHEMATICS-20SC01T

UNIT : 1 - MATRICES AND DETERMINANTS

Matrices

Matrix: A matrix is a rectangular arrangement of a numbers in a rows and columns with in a closed brackets called as matrix.

Matrices is a plural form of matrix.

Eg: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

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Order of a matrix:

If number of rows and columns are represented by m and n then order of a matrix can be defined as $m \times n$

Types of matrices:

1. Square matrix : A matrix in which the number of rows is equal to the number of columns ,is called a square matrix .e.g.: $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$
2. Row matrix : A matrix in which it having only one row but many number of columns is called row matrix .e.g.: $\begin{bmatrix} 5 & 4 \end{bmatrix}$
3. Column matrix: A matrix having only one column but many number of rows is called column matrix e.g.: $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
4. Diagonal matrix :It's a square matrix in which the non principal diagonal elements are equal to zero .e.g.: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
5. Scalar matrix : It's a diagonal matrix in which the principal diagonal elements are same other than 1. e.g.: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
6. Identity (unit) matrix : It's a scalar matrix in which the principal diagonal elements are equal to 1. .e.g.: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
7. Null matrix : A matrix in which all the elements are zero is called as null (zero) matrix .e.g.: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8. Symmetric matrix : A square matrix is said to be symmetric if it remains same when rows are changed into columns or columns are changed into rows .e.g.: $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ $A^1 = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

9. Equal matrix : If two matrices are said to be equal

   i) They have the same order and

   ii) Corresponding elements are equal

Eg: $\begin{bmatrix} x & y \\ w & z \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$ if x=5, y=3, w=2, z=4

Algebra of matrices:

Transpose of a matrices:

The matrix obtained by interchanging rows into columns or columns into rows is called transpose of a matrix and it is represented by $A^1(or)A^T$.

e.g.: $A = [1 \quad 2]_{1\times 2}$ $A^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1}$

Addition of matrices

If A and B are two matrices of same order $m\times n$ then their sum $A+B$ is also a matrix of order $m\times n$ and is obtained by adding corresponding elements of A and B.

Examples: (1) If $X = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and $Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $X+Y = \begin{bmatrix} w+a & x+b \\ y+c & z+d \end{bmatrix}$

(2) If $A = \begin{bmatrix} 5 & 3 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 0 & 1 \\ 1 & 5 \end{bmatrix}$ then $A+B = \begin{bmatrix} 5+3 & 3+4 \\ 1+0 & 2+1 \\ 2+1 & 4+5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 1 & 3 \\ 3 & 9 \end{bmatrix}$

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Subtraction of matrices

If A and B are two matrices of same order $m \times n$ then their difference $A - B$ is also a matrix of order $m \times n$ and is obtained by subtracting elements of B from corresponding elements of A.

Examples: (1) If $X = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ and $Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $X - Y = \begin{bmatrix} w-a & x-b \\ y-c & z-d \end{bmatrix}$

(2) If $A = \begin{bmatrix} 5 & 3 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 0 & 2 \\ 5 & 1 \end{bmatrix}$ then $A - B = \begin{bmatrix} 5-3 & 3-4 \\ 1-0 & 2-2 \\ 2-5 & 4-1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 3 \end{bmatrix}$

Scalar Multiplication of a matrix

If A is a matrix and k is a scalar then the scalar multiplication kA is a matrix obtained by multiplying each elements of A by scalar k.

Examples: (1) If $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ then $kA = \begin{bmatrix} kw & kx \\ ky & kz \end{bmatrix}$

(2) If $B = \begin{bmatrix} 4 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ then $2B = \begin{bmatrix} 8 & 2 & 0 \\ -2 & 4 & 6 \end{bmatrix}$

PROBLEMS

Example 1: If $A = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$ then find $4A$.

Solution: Given $A = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow$ $4A = 4\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow$ $4A = \begin{bmatrix} 4\times 1 & 4\times 3 \\ 4\times 4 & 4\times(-2) \end{bmatrix}$

$\Rightarrow$ $4A = \begin{bmatrix} 4 & 12 \\ 16 & -8 \end{bmatrix}$

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Example 2: If $B = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \end{bmatrix}$ then find $2B$.

Solution: Given $B = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \end{bmatrix}$

$$\Rightarrow \quad 2B = 2\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \quad 2B = \begin{bmatrix} 2 & 0 & -6 \\ 6 & 4 & 8 \end{bmatrix}$$

Example 3: If $X = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 0 \\ 4 & 7 \end{bmatrix}$ then find $X+Y$.

Solution: Given $X = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 0 \\ 4 & 7 \end{bmatrix}$

Consider $X + Y = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & 7 \end{bmatrix}$

$$\Rightarrow \quad X + Y = \begin{bmatrix} 3+2 & 1+0 \\ 2+4 & 5+7 \end{bmatrix}$$

$$\Rightarrow \quad X + Y = \begin{bmatrix} 5 & 1 \\ 6 & 12 \end{bmatrix}$$

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Example 4: If $A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 3 & 7 \\ 0 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \\ 7 & 5 & -1 \\ 2 & 6 & 4 \end{bmatrix}$ then find $A+B$.

Solution: Given $A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 3 & 7 \\ 0 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \\ 7 & 5 & -1 \\ 2 & 6 & 4 \end{bmatrix}$

Consider $A + B = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 3 & 7 \\ 0 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ 7 & 5 & -1 \\ 2 & 6 & 4 \end{bmatrix}$

$$\Rightarrow \quad A + B = \begin{bmatrix} 1+0 & 2+2 & 6+1 \\ 5+7 & 3+5 & 7+(-1) \\ 0+2 & (-1)+6 & 4+4 \end{bmatrix}$$

$$\Rightarrow \quad A + B = \begin{bmatrix} 1 & 4 & 7 \\ 12 & 8 & 6 \\ 2 & 5 & 8 \end{bmatrix}$$

Example 5: If $A=\begin{bmatrix}2&1\\6&2\\4&3\end{bmatrix}$ and $B=\begin{bmatrix}5&-1\\2&0\\1&4\end{bmatrix}$ then find $3B-2A$.

Solution: Given $A=\begin{bmatrix}2&1\\6&2\\4&3\end{bmatrix}$ and $B=\begin{bmatrix}5&-1\\2&0\\1&4\end{bmatrix}$

Consider $3B-2A=3\begin{bmatrix}5&-1\\2&0\\1&4\end{bmatrix}-2\begin{bmatrix}2&1\\6&2\\4&3\end{bmatrix}$

$$=\begin{bmatrix}15&-3\\6&0\\3&12\end{bmatrix}-\begin{bmatrix}4&2\\12&4\\8&6\end{bmatrix}$$

$$=\begin{bmatrix}15-4&-3-2\\6-12&0-4\\3-8&12-6\end{bmatrix}$$

$$3B-2A=\begin{bmatrix}11&-5\\-6&-4\\-5&6\end{bmatrix}$$

Example 6: If $\begin{bmatrix}1&2\\4&x\end{bmatrix}+\begin{bmatrix}y&3\\2&5\end{bmatrix}=\begin{bmatrix}7&5\\6&5\end{bmatrix}$ then find values of x and y.

Solution: Given $\begin{bmatrix}1&2\\4&x\end{bmatrix}+\begin{bmatrix}y&3\\2&5\end{bmatrix}=\begin{bmatrix}7&5\\6&5\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}1+y&2+3\\4+2&x+5\end{bmatrix}=\begin{bmatrix}7&5\\6&5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1+y&5\\6&x+5\end{bmatrix}=\begin{bmatrix}7&5\\6&5\end{bmatrix}$$

Two matrices are said to be equal if they are of same order and their corresponding elements are equal.

By equality of matrices, we get

$$1+y=7 \quad \text{And} \quad x+5=5$$

$$\Rightarrow \quad y=7-1 \text{ And } \quad x=5-5$$

$$\Rightarrow \quad y=6 \quad \text{And} \quad x=0$$

Example 7: If $A+B=\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $A-B=\begin{bmatrix} 1 & 5 \\ 4 & -6 \end{bmatrix}$ then find A.

Solution: Given $A+B=\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ --- (1)

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and $A-B=\begin{bmatrix} 1 & 5 \\ 4 & -6 \end{bmatrix}$ --- (2)

Adding (1) and (2), we get

$$(A+B)+(A-B)=\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}+\begin{bmatrix} 1 & 5 \\ 4 & -6 \end{bmatrix}$$

$$\Rightarrow \quad A+B+A-B=\begin{bmatrix} 2+1 & -1+5 \\ 3+4 & 4-6 \end{bmatrix}$$

$$\Rightarrow \quad 2A=\begin{bmatrix} 3 & 4 \\ 7 & -2 \end{bmatrix}$$

$$\Rightarrow \quad A=\frac{1}{2}\begin{bmatrix} 3 & 4 \\ 7 & -2 \end{bmatrix}$$

$$\Rightarrow \quad A=\begin{bmatrix} 3/2 & 2 \\ 7/2 & -1 \end{bmatrix}$$

ASSIGNMENT PROBLEMS

1. If $A=\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ then find $3A$.

2. If $B=\begin{bmatrix} 3 & 0 & 2 \\ -1 & 4 & 5 \end{bmatrix}$ then find $2B$.

3. If $A=\begin{bmatrix} 3 & 6 \\ 15 & 9 \\ 12 & 3 \end{bmatrix}$ then find $\frac{1}{3}A$.

4. Find $2X$ given that $X=\begin{bmatrix} 1 & 0 & 3 \\ -2 & 5 & 7 \\ 1 & 6 & 4 \end{bmatrix}$.

5. If $X=\begin{bmatrix} 3 & 5 & -1 \\ 6 & 2 & 4 \\ 1 & 7 & 0 \end{bmatrix}$ and $Y=\begin{bmatrix} 2 & 3 & 1 \\ 4 & -1 & 0 \\ 7 & 5 & 1 \end{bmatrix}$ then find $X+Y$.

6. If $A = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 3 & 6 \end{bmatrix}$ then find $2A + B$.

7. If $A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$ then find $3A - B$.

8. If $A = \begin{bmatrix} 1 & 5 \\ 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$ then verify that $A + B = B + A$.

9. If $\begin{bmatrix} x & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ -1 & y \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 2 & 7 \end{bmatrix}$ then find the values of x and y.

10. If $A + B = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ then find matrix A and B

Multiplication of matrices:

If matrix A of order $m \times n$ and matrix B of order $n \times p$ then AB is defined as the matrix of order $m \times p$ it is obtained by multiplying by elements of 1st row of matrix A by corresponding elements of 1st column of matrix B and adding there products .

To perform **multiplication of two matrices**, we should make sure that the number of columns in the 1st matrix is equal to the rows in the 2nd matrix. Therefore, the resulting matrix product will have a number of rows of the 1st matrix and a number of columns of the 2nd matrix. The order of the resulting matrix is the **matrix multiplication order**.

E.g. let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ $B = \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}$ www.mathswithme.in

Then $AB = \begin{bmatrix} ag + bi + ck & ah + bj + cl \\ dg + ei + fk & dh + ej + fl \end{bmatrix}$

Problems

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ then find the product of two matrices.

Soln: $AB = \begin{bmatrix} 1 \times 2 + 2 \times 3 & 1 \times 1 + 2 \times 4 \\ 3 \times 2 + 4 \times 3 & 3 \times 1 + 4 \times 4 \end{bmatrix}$

$AB = \begin{bmatrix} 2 + 6 & 1 + 8 \\ 6 + 12 & 3 + 16 \end{bmatrix}$

$AB = \begin{bmatrix} 8 & 9 \\ 18 & 19 \end{bmatrix}$ order of AB is 2×2

2.

DETERMINANT

Definition:

A determinant is a real number associated with every square matrix.

The determinant of a square matrix A is denoted by "det A" or | A |.

Example: If A = $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, then det A = $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$

Expansion of Determinant of a 2×2 Matrix:

IF A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then det A = $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$|\mathbf{A}| = \mathbf{ad - bc}$$

Example: 1) Let A $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix}$

= (1) (8) - (2) (3)

= 8 – 6

$|\mathbf{A}| = \mathbf{2}$.

2) Let B = $\begin{bmatrix} 3 & 5 \\ 1 & -4 \end{bmatrix}$ then $|B| = \begin{vmatrix} 3 & 5 \\ 1 & -4 \end{vmatrix}$

= (3) (-4) – (1) (5)

= -12-5

$|B| = -17$

Expansion of Determinant of a 3×3 Matrix

Let C = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Then, $|\boldsymbol{C}| = \boldsymbol{a_1} \begin{vmatrix} \boldsymbol{b_2} & \boldsymbol{c_2} \\ \boldsymbol{b_3} & \boldsymbol{c_3} \end{vmatrix} - \boldsymbol{b_1} \begin{vmatrix} \boldsymbol{a_2} & \boldsymbol{c_2} \\ \boldsymbol{a_3} & \boldsymbol{c_3} \end{vmatrix} + \boldsymbol{c_1} \begin{vmatrix} \boldsymbol{a_2} & \boldsymbol{b_2} \\ \boldsymbol{a_3} & \boldsymbol{b_3} \end{vmatrix}$

Note: **Sign rule for expansion of order 3 determinant** $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Example:

Let A = $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 8 & 1 \\ 3 & 0 & 2 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 8 & 1 \\ 3 & 0 & 2 \end{vmatrix}$

Value of $|A| = 2\begin{vmatrix} 8 & 1 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 5 & 1 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 5 & 8 \\ 3 & 0 \end{vmatrix}$

= 2(16 – 0) – 3(10 – 3) + 4(0 – 24)

=2 (16) – 3(7) + 4(– 24)

$= 32 - 21 - 96$

$|A| = -85$

SINGULAR AND NON-SINGULAR MATRIX

Definition

A square matrix B is said to be singular if its determinant value is zero, i.e. $|B| = 0$, otherwise non -singular.

Example: 1) Let B= $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ then

$|B| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$

$= (1)(6) - (2)(3)$

$= 6 - 6 = 0$

$|B|$ **= Zero Therefore B is singular.**

2) Let A = $\begin{bmatrix} 10 & 20 \\ 2 & 4 \end{bmatrix}$ then

$|A| = \begin{vmatrix} 10 & 20 \\ 2 & 4 \end{vmatrix}$

$= (10)(4) - (2)(20)$

$= 40 - 40$

$|A|$ **= 0, therefore A is singular.**

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3) Let A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then

$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

$= 2(0 - 0) - 0(0 - 0) + 0(0 - 5)$

$|A|$ **= 0 Therefore A is singular.**

SOLVED PROBLEMS

I. Check whether the following matrices are singular:

1) A = $\begin{bmatrix} 5 & 25 \\ 1 & 5 \end{bmatrix}$

Soln: $|A| = \begin{vmatrix} 5 & 25 \\ 1 & 5 \end{vmatrix}$

$= (5)(5) - (1)(25)$

$= 25 - 25$

$|A| = 0$ **therefore A is singular matrix.**

2) $A = \begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$

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Soln: $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 5 \end{vmatrix}$

$= (2)(5) - (1)(5)$

$= 10 - 5$

$|A| = 5 \neq 0$ Therefore A is non-singular matrix.

3) $A = \begin{bmatrix} 4 & 12 \\ 2 & 6 \end{bmatrix}$

Soln: $|A| = \begin{vmatrix} 4 & 12 \\ 2 & 6 \end{vmatrix}$

$= (4)(6) - (12)(2)$

$= 24 - 24$

$|A| = 0$ therefore A is singular matrix.

4) $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 7 & 5 & 6 \end{bmatrix}$

Soln: $|A| = \begin{vmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 7 & 5 & 6 \end{vmatrix}$

$= 3(12-20) - 1(36-28) + 2(30-14)$

$= 3(-8) - 1(8) + 2(16)$

$= -24 - 8 + 32$

$= -32 + 32$

$|A| = 0$ therefore A is singular matrix.

II Find the value of x

5) If $\begin{vmatrix} 1 & 5 & 7 \\ 2 & x & 14 \\ 3 & 1 & 2 \end{vmatrix} = 0$

Soln: Given that $\begin{vmatrix} 1 & 5 & 7 \\ 2 & x & 14 \\ 3 & 1 & 2 \end{vmatrix} = 0$

Expanding the determinant, we get,

$1(2x - 14) - 5(4 - 42) + 7(2 - 3x) = 0$

$2x - 14 - 5(-38) + 14 - 21x = 0$

$2x - 14 + 190 + 14 - 21x = 0$

$-19x + 190 = 0$

$19x = 190$

$x = \frac{190}{19}$

$x = 10$

6) If A = $\begin{bmatrix} 1 & 2 & 3 \\ x & 4 & 1 \\ 3 & 6 & 5 \end{bmatrix}$ is Singular Matrix

Soln: Given A is singular i.e. |A| = 0

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ x & 4 & 1 \\ 3 & 6 & 5 \end{vmatrix} = 0$$

Expanding the determinant, we get,

1 (20 – 6) – 2 (5 x – 3) + 3(6 x – 12) = 0

1(14) -2 (5 x -3) +3(6 x -12) = 0

14-10 x +6+18 x -36 =0

-16+8 x = 0

8 x = 16

$x = \frac{16}{8}$

$\mathbf{x = 2}$

I) EVALUATE THE FOLLOWING

1) $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$ (2) $\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

(4) $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix}$ (5) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

(II) FIND THE VALUE OF 'x' IN THE FOLLOWING

1) $\begin{vmatrix} x & -1 \\ 2 & 1 \end{vmatrix}$=0 (2) $\begin{vmatrix} x & 8 \\ 8 & x \end{vmatrix}$=0

(3) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & x \\ 7 & 8 & 9 \end{vmatrix}$=0 (4) $\begin{vmatrix} 1 & 2 & 9 \\ 2 & x & 0 \\ 3 & 7 & -6 \end{vmatrix}$=0

(5) $\begin{vmatrix} 2 & x-1 & -3 \\ 1 & -2 & 4 \\ 3 & -1 & 5 \end{vmatrix}$=3x-1.

1.5 : CRAMER'S RULE-TWO VARIABLES

SOLUTION OF SYSTEM OF *TWO* SIMULTANEOUS LINEAR EQUATIONS BY CRAMER'S RULE (DETERMINANT METHOD)

Consider the system of equations:

$$a_1\, x + b_1 y = c_1 \quad \text{and} \quad a_2\, x + b_2 y = c_2$$

$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ (det of coefficient)

$\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1 b_2 - c_2 b_1$ (det obtained by replacing 1st column f Δ by constantsc_1, c_2)

$\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1 c_2 - a_2$ (det obtained by replacing 2nd column of Δ by constantsc_1, c_2)

Therefore, $x = \frac{\Delta_1}{\Delta}$ and $y = \frac{\Delta_2}{\Delta}$. Provaided ($\Delta \neq 0$)

- Cramer's Rule can also be used to solve the simultaneous equations of 'n' variables

Example

Solve 2x +3y =1; 3x –y = – 2 by Cramer's rule.

Soln: $\Delta = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (3)(3) = -2 - 9 = -11$

$\Delta_1 = \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix} = (1)(-1) - (-2)(3) = 1+6 = 5$

$\Delta_2 = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = (2)(-2) - (1)(3) = -4 - 3 = -7$

Therefore, $\boldsymbol{x = \frac{\Delta_1}{\Delta} = \frac{5}{-11}}$ and $\boldsymbol{y = \frac{\Delta_2}{\Delta} = \frac{-7}{-11} = \frac{7}{11}}$

Worked Examples on Cramer's Rule

1) Solve the equations 2x + y =1; 3x + 2y =1 by method of determinants.

Soln: Given system of equation is 2x + y =1

3x +2y =1

$let\ \Delta = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$

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$\Delta_1 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$

$\Delta_2 = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2 - 3 = -1$

$now\ x = \frac{\Delta_1}{\Delta} = \frac{1}{1} = 1\ and\ y = \frac{\Delta_2}{\Delta} = \frac{-1}{1} = -1$

$\therefore \boldsymbol{x = 1\ and\ y = -1}$

2) Solve the equations x + y = 3; 2x + 3y = 8 by Cramer's rule

Soln: Given system of equation is x + y = 3

2x + 3y = 8

$let\ \Delta = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1;$

$\Delta_1 = \begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix} = 9 - 8 = 1 \qquad and$

$\Delta_2 = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 8 - 6 = 2$

$now\ x = \frac{\Delta_1}{\Delta} = \frac{1}{1} = 1\ and\ y = \frac{\Delta_2}{\Delta} = \frac{2}{1} = 2$

$\therefore \boldsymbol{x = 1\ and\ y = -1}$

3) Solve 2x - 3y = 5, 7x – y = 8 by Cramer's rule.

Soln: Given system of the equations is 2x – 3y = 5

7x – y = 8

$let\quad \Delta = \begin{vmatrix} 2 & -3 \\ 7 & -1 \end{vmatrix} = -2 + 21 = 19$

$\Delta_1 = \begin{vmatrix} 5 & -3 \\ 8 & -1 \end{vmatrix} = -5 + 2 = 19$

$\Delta_2 = \begin{vmatrix} 2 & 5 \\ 7 & 8 \end{vmatrix} = 16 - 35 = -19$

$$now\ x = \frac{\Delta_1}{\Delta} = \frac{19}{19} = 1\ and\ y = \frac{\Delta_2}{\Delta} = \frac{-19}{19} = -1$$

$$\therefore \boldsymbol{x = 1\ and\ y = -1}$$

4) Solve 3u + 4v = 10; 2u – 3v = 1 by Cramer's rule

Soln: Given system of equation is $3u + 4v = 10$

$2u - 3v = 1$

$$let\ \Delta = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix} = -9 - 8 = -17;$$

$$\Delta_1 = \begin{vmatrix} 10 & 4 \\ 1 & -3 \end{vmatrix} = -30 - 4 = -34$$

$$\Delta_2 = \begin{vmatrix} 3 & 10 \\ 2 & 1 \end{vmatrix} = 3 - 20 = -17$$

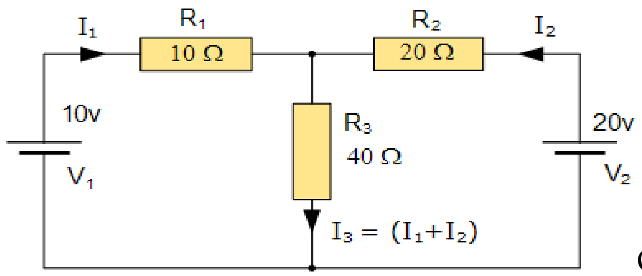
$$now\ u = \frac{\Delta_1}{\Delta} = \frac{-34}{-17} = 2\ and\ u = \frac{\Delta_2}{\Delta} = \frac{-17}{-17} = 1$$

$$\therefore u = \mathbf{2\ and}\ v = \mathbf{1}$$

Applications of Cramer's rule to mesh analysis

The following is an example to demonstrate the application of crammer's rule to solve mesh current analysis problems;

Mesh Current Analysis Circuit



One simple method of reducing the amount of maths involved is to

Analyse the circuit using Kirchhoff's Current Law equations to determine the currents, I_1 and I_2 flowing in the two resistors. Then there is no need to calculate the current I_3 as it is just the sum of I_1 and I_2. So Kirchhoff's second voltage law simply becomes:

- Equation No 1 : $10 = 50I_1 + 40I_2$
- Equation No 2 : $20 = 40I_1 + 60I_2$

The above equations can be solved using crammer's rule, resulting in detection of I_1 and I_2.

EXERCISE www.mathswithme.in

Solve by Cramer's Rule:

1) $3x+y = 4$, $x +3y = 4$

(2) $2x -3y =5$, $7x -y = 8$

(3) $5x+3y =1$, $3x +5y = -9$

(4) $Y=4$, $x+3y=4$

(5) $3x +4y-7=0$, $7x-y-6=0$

(6) $R_1+ 4R_2 =70$, $2R_2 -3R_1 = 0$

1.6 : CRAMER'S RULE-THREE VARIABLES

SOLUTION OF SYSTEM OF *THREE* SIMULTANEOUS LINEAR EQUATIONS BY CRAMER'S RULE (DETERMINANT METHOD)

Consider the equations $a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$,

In three variables x, y and z.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(det of coefficient)}$$

$$\Delta_1 = \begin{vmatrix} \boldsymbol{d_1} & b_1 & c_1 \\ \boldsymbol{d_2} & b_2 & c_2 \\ \boldsymbol{d_3} & b_3 & c_3 \end{vmatrix} \quad \text{(det obtained by replacing 1}^{\text{st}}\text{ column of } \Delta \text{ by constants)}$$

$$\Delta_2 = \begin{vmatrix} a_1 & \boldsymbol{d_1} & c_1 \\ a_2 & \boldsymbol{d_2} & c_2 \\ a_3 & \boldsymbol{d_3} & c_3 \end{vmatrix} \quad \text{(det obtained by replacing 2}^{\text{nd}}\text{ column of } \Delta \text{ by constants)}$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & \boldsymbol{d_1} \\ a_2 & b_2 & \boldsymbol{d_2} \\ a_3 & b_3 & \boldsymbol{d_3} \end{vmatrix} \quad \text{(det obtained by replacing 3}^{\text{rd}}\text{ column of } \Delta \text{ by constants)}$$

Therefore, $x=\frac{\Delta_x}{\Delta}$, $y=\frac{\Delta_y}{\Delta}$ and $z=\frac{\Delta_z}{\Delta}$ provided $\Delta \neq 0$

Example:

Solve using cramer's rule: 5x – 2y – 3z =17 , 3x – y+z =15 and x+y – 6z = – 13

Solution: Given system Equation of

5x – 2y – 3z =17

3x – y+z =15

x+y – 6z = – 13

then

$$\Delta=\begin{vmatrix}5 & -2 & -3\\3 & -1 & 1\\1 & 1 & -6\end{vmatrix}$$

=5(6 – 1)+ 2(– 18 – 1) – 3(3+1) = 25 – 38 – 12 = – 25.

$$\Delta_1=\begin{vmatrix}\mathbf{17} & -2 & -3\\\mathbf{15} & -1 & 1\\\mathbf{-13} & 1 & -6\end{vmatrix}$$

= 17(6 – 1)+ 2(– 90+13) – 3(15 – 13) = 85 – 154 – 6 = – 75.

$$\Delta_2=\begin{vmatrix}5 & \mathbf{17} & -3\\3 & \mathbf{15} & 1\\1 & \mathbf{-13} & -6\end{vmatrix}$$

=5(– 90+13) – 17(– 18 – 1) – 3(– 39 – 15)

= – 385+323+162=100.

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$$\Delta_3=\begin{vmatrix}5 & -2 & \mathbf{17}\\3 & -1 & \mathbf{15}\\1 & 1 & \mathbf{-13}\end{vmatrix}$$

=5(13 – 15)+ 2(– 39 – 15)+17(3+1)

= – 10 – 108+68= – 50.

$x=\frac{\Delta_x}{\Delta}=\frac{-75}{-25}=3,\quad y=\frac{\Delta_y}{\Delta}=\frac{100}{-25}=-4,\quad z=\frac{\Delta_z}{\Delta}=\frac{-50}{-25}=2.$

Therefore, **x=3 , y=-4 and z=2**

Problems on 3 linear equations using cramer's rule:

1)solve the following equation using Cramer's rule

x +y +z = 7;   2x +3y +2z = 17;   4x +9y +z = 37

Solution : Given system of equation is

x +y +z=7

2x +3y +2z=17

4x +9y +z=37

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 9 & 1 \end{vmatrix}$= 1 (3-18)-1 (2-8)+1 (18-12)= -15+6+6= -3

$\Delta_1 = \begin{vmatrix} 7 & 1 & 1 \\ 17 & 3 & 2 \\ 37 & 9 & 1 \end{vmatrix}$=7(3-18)-1(17-74)+1(153-111)

= -105+57+42= -6

$\Delta_2 = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 17 & 2 \\ 4 & 37 & 1 \end{vmatrix}$1(17-74)-7(2-8)+1(74-68)= -57+42+6= -9

$\Delta_3 = \begin{vmatrix} 1 & 1 & 7 \\ 2 & 3 & 17 \\ 4 & 9 & 37 \end{vmatrix}$=1(111-153)-1(74-68)+7(18-12)

= -42-6+42= -6

$$now\ x = \frac{\Delta_1}{\Delta} = \frac{-6}{-3} = 2\,;\mathrm{y} = \frac{\Delta_2}{\Delta} = \frac{-9}{-3} = 3\ and\ z = \frac{\Delta_3}{\Delta} = \frac{-6}{-3} = 2$$

$$\therefore x = 2,\ y = 3,\ z = 2$$

2)solve the following equation by Cramer's rule

2x +y –z = 3; x +y +z = 1; x -2y -3z = 4

Solution :Given system of equation is 2x +y –z = 3;

x +y +z = 1

x -2y -3z = 4

Let $\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix}$ =2(-3+2)-1(-3-1)-1(-2-1)=-2+4+3 = 5

$\Delta_1 = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix}$=3(-3+2)-1(-3-4)-1(-2-4)= -3+7+6= 10

$\Delta_2 = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix}$=2(-3-4)-3(-3-1)-1(4-1)= -14+12-3= -5

$\Delta_3 = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix}$=2(4+2)-1(4-1)+3(-2-1)= 12-3-9= 0

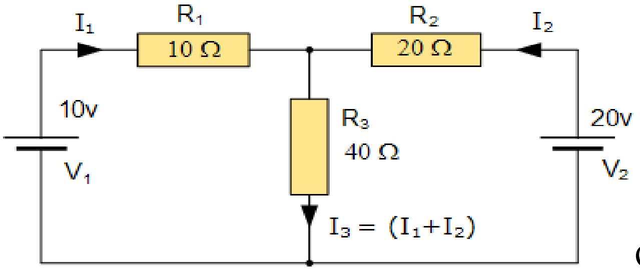
$$now\ x = \frac{\Delta_1}{\Delta} = \frac{100}{5} = 2; y = \frac{\Delta_2}{\Delta} = \frac{-5}{5} = -1\ and\ z = \frac{\Delta_3}{\Delta} = \frac{0}{5} = 0$$

$$\therefore x = 2,\ y = -1,\ z = 0$$

Applications of cramer's rule to mesh analysis

The following is an example to demonstrate the application of cramer's rule to solve mesh current analysis problems;

Mesh Current Analysis Circuit



One simple method of reducing the amount of math's involved is to

analyse the circuit using Kirchhoff's Current Law equations to determine the currents, I_1 and I_2 flowing in the two resistors. Then there is no need to calculate the current I_3 as its just the sum of I_1 and I_2. So Kirchhoff's second voltage law simply becomes:

- Equation No 1 : $10 = 50I_1 + 40I_2$
- Equation No 2 : $20 = 40I_1 + 60I_2$

The above equations can be solved using cramer's rule, resulting in detection of I_1 and I_2.

The following are the problems for practice:

I)<u>Evaluate the following</u> :

1) $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$ (2) $\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

(4) $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix}$ (5) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

(II)<u>Find the value of 'x' in the following</u>:

1) $\begin{vmatrix} x & -1 \\ 2 & 1 \end{vmatrix}$=0 (2) $\begin{vmatrix} x & 8 \\ 8 & x \end{vmatrix}$=0 (3) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & x \\ 7 & 8 & 9 \end{vmatrix}$=0 (4) $\begin{vmatrix} 1 & 2 & 9 \\ 2 & x & 0 \\ 3 & 7 & -6 \end{vmatrix}$=0 (5) $\begin{vmatrix} 2 & x-1 & -3 \\ 1 & -2 & 4 \\ 3 & -1 & 5 \end{vmatrix}$=3x-

1.

(III) solve by cramer's rule:

1) $3x+y=4$, $x+3y=4$

(2) $2x-3y=5$, $7x-y=8$

(3) $5x+3y=1$, $3x+5y=-9$

4) $x+y+z=7$, $2x+3y+2z=17$, $4x+9y+z=37$.

5) $x+y+z=7$, $x+2y+3z=16$, $x+3y+4z=22$

6) $2x+y=1$, $y+2z=7$, $3z-2x=11$

(IV) a) Find the value of R_1 and R_2 cramer's rule:

$R_1+4R_2=70$, $2R_2-3R_1=0$

b)In an electrical network, currents i_1, i_2 , i_3 are given by,

$3i_1+i_2+i_3=8$,

$2i_1-3i_2-2i_3=-5$,

$7i_1+2i_2-5i_3=0$.

Calculate the current i_2 using cramer's rule.

Mcq's:

1)Which among the following matrices can be expanded as a determinant.

(a) $\begin{bmatrix} 1 & 2 & 4 \\ 6 & 3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 5 & 7 \end{bmatrix}$

Ans: option (b).

2)which among the following statements is true?

(a) A det is a square matrix of order n xn.

(b) A det is a real number associated with a square matrix.

(c)A det is always singular.

(d)A det is a matrix with equal number of rows and columns.

Ans: option (b).

3) The det value of $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ is

(a) 3 (b) 15 (c) 30 (d) 10.

Ans: option (c).

4) If matrix ' A ' is singular matrix , then

(a) $|A|=100$ (b) $|A|=0$ (c) $|A|$ is plural (d) $|A|$ is any real number

Ans: option (b)

5) If A = $\begin{bmatrix} 1 & 2 \\ 3 & x \end{bmatrix}$ is singular , then x=

(a) 6 (b) -6 (c) 16 (d) 0.

Ans: option (a).

MCQ'S ON CRAMER'S RULE

1)Which among the following is a method to solve system of simultaneous equations:

(a)chain rule (b) Pythagoras rule

(c) cramer's rule (d) rule of matrix

soln: option (c).

2) To find value of 'x' in the system of equations : x+ y =1 and 2x +3y =2 how many determinants are required in Cramer's rule

(a)1 (b) 2 (c) 3 (d) 4

Soln: option (b)

3) To find value of 'x' and 'y' in the system of equations :

x+ y =1 and 2x +3y =2

how many determinants are required in Cramer's rule

(a)1 (b) 2 (c) 3 (d) 4

Soln: option (c)

4) The solution set of linear equations: x+ y =1 and 2x +3y =2 is

(a)x=1, y=0 (b) x=0, y=1 (c) x=-1, y=0 (d) x=0, y=0

Soln: option (a) $\Delta = 1$, $\Delta_1 = 1$, $\Delta_2 = 0$

$$x = \frac{\Delta_1}{\Delta} = \frac{1}{1} = 1 \text{ and } y = \frac{\Delta_2}{\Delta} = \frac{0}{1} = 0$$

5)The value of 'x' that satisfies the system of equations

3x+4y=7 and 7x -y=6 is

(a)x=-2 (b) x=10 (c) x=1 (d) x=0

Soln: option (c) $\Delta = -31$, $\Delta_1 = -31$, $x = \frac{\Delta_1}{\Delta} = \frac{1}{1} = 1$

Consider the system of equations:

$$a_1 x + b_1 y = c_1 \quad \text{and} \quad a_2 x + b_2 y = c_2$$

$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ (det of coefficient)

$\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1 b_2 - c_2 b_1$ (det obtained by replacing 1st column f Δ by

constantsc_1, c_2)

$\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2$ (det obtained by replacing 2nd column of Δ by constants c_1, c_2)

Therefore, $x = \frac{\Delta_1}{\Delta}$ and $y = \frac{\Delta_2}{\Delta}$. Provaided ($\Delta \neq 0$)

- Cramer's Rule can also be used to solve the simultaneous equations of 'n' variables

Example

Solve 2x +3y =1 ; 3x –y = – 2 by Cramer's rule.

Soln : $\Delta = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}$ = (2)(– 1) – (3)(3)= – 2 – 9 = – 11

$\Delta_1 = \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}$ = (1)(– 1) – (– 2)(3)= 1+6 = 5

$\Delta_2 = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$ = (2)(– 2) – (1)(3)= – 4 – 3 = – 7

Therefore, $\boldsymbol{x = \frac{\Delta_1}{\Delta} = \frac{5}{-11}}$ and $\boldsymbol{y = \frac{\Delta_2}{\Delta} = \frac{-7}{-11} = \frac{7}{11}}$

Worked Examples on Cramer's Rule

1) Solve the equations 2x + y =1; 3x + 2y =1 by method of determinants.

Soln: Given system of equation is 2x + y =1

3x +2y =1

let $\Delta = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$

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$\Delta_1 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$

$\Delta_2 = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2 - 3 = -1$

$$now\ x = \frac{\Delta_1}{\Delta} = \frac{1}{1} = 1\ and\ y = \frac{\Delta_2}{\Delta} = \frac{-1}{1} = -1$$

$\therefore \boldsymbol{x = 1\ and\ y = -1}$

2) Solve the equations x + y = 3; 2x + 3y = 8 by Cramer's rule

Soln: Given system of equation is x + y = 3

2x + 3y = 8

$$let\ \Delta = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1;$$

$$\Delta_1 = \begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix} = 9 - 8 = 1 \quad and$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 8 - 6 = 2$$

$$now\ x = \frac{\Delta_1}{\Delta} = \frac{1}{1} = 1\ and\ y = \frac{\Delta_2}{\Delta} = \frac{2}{1} = 2$$

$\therefore \boldsymbol{x = 1\ and\ y = -1}$

3) Solve 2x - 3y = 5, 7x – y = 8 by Cramer's rule.

Soln: Given system of the equations is 2x – 3y = 5

7x – y = 8

$$let\ \ \Delta = \begin{vmatrix} 2 & -3 \\ 7 & -1 \end{vmatrix} = -2 + 21 = 19$$

$$\Delta_1 = \begin{vmatrix} 5 & -3 \\ 8 & -1 \end{vmatrix} = -5 + 2 = 19$$

$$\Delta_2 = \begin{vmatrix} 2 & 5 \\ 7 & 8 \end{vmatrix} = 16 - 35 = -19$$

$$now\ x = \frac{\Delta_1}{\Delta} = \frac{19}{19} = 1\ and\ y = \frac{\Delta_2}{\Delta} = \frac{-19}{19} = -1$$

$\therefore \boldsymbol{x = 1\ and\ y = -1}$

4) Solve 3u + 4v = 10; 2u – 3v = 1 by Cramer's rule

Soln: Given system of equation is $3\,u + 4\,v = 10$

$$2\,u - 3\,v = 1$$

$$let\ \Delta = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix} = -9 - 8 = -17;$$

$$\Delta_1 = \begin{vmatrix} 10 & 4 \\ 1 & -3 \end{vmatrix} = -30 - 4 = -34$$

$$\Delta_2 = \begin{vmatrix} 3 & 10 \\ 2 & 1 \end{vmatrix} = 3 - 20 = -17$$

$$now\ u = \frac{\Delta_1}{\Delta} = \frac{-34}{-17} = 2\ and\ u = \frac{\Delta_2}{\Delta} = \frac{-17}{-17} = 1$$

$$\therefore u = \mathbf{2\ and}\ v = \mathbf{1}$$

EXERCISE

Solve by Cramer's Rule:

1) 3x+y = 4, x +3y = 4

(2) 2x -3y =5 , 7x –y = 8

(3) 5x+3y =1 , 3x +5y = -9

(4) y=4, x+3y=4

(5) 3x +4y-7=0, 7x-y-6=0

(6) R_1+ $4R_2$ =70, $2R_2$ -$3R_1$ = 0

Minor of an element of a matrix: Minor of an element $\mathbf{a_{ij}}$ of a matrix is the determinant obtained by deleting $\boldsymbol{i^{th}}$ row and $\boldsymbol{j^{th}}$ column of matrix.

Steps to find Minor of an element

For each element of the matrix

Step (1): Delete the elements on the current row and column

Step (2): Calculate the determinant of remaining elements

Example 1: Consider a 2×2 matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

Follow the same method to find the Minor of remaining

To find Minor of a_1

Step (1): $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

Minor of a_1 = $|\ b_2\ | = b_2$

Minor of a_2 = $|\ b_1\ | = b_1$

Minor of $b_1 = | a_2 | = a_2$

Minor of $b_2 = | a_1 | = a_1$

Example 2: Consider a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Minor of $a_1 =$

To find Minor of a_1

Step (1): $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$

Follow the same method to find the Minor of remaining elements

Step (2): $\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)$

$\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)$

Minor of $a_2 = \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = (b_1c_3 - b_3c_1)$

Minor of $a_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)$

Minor of $b_1 = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = (a_2c_3 - a_3c_2)$

Minor of $b_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = (a_1c_3 - a_3c_1)$

Minor of $b_3 = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = (a_1c_2 - a_2c_1)$

Minor of $c_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)$

Minor of $c_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = (a_1b_3 - a_3b_1)$

Minor of $c_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = (a_1b_2 - a_2b_1)$

Problems on finding minor of an element of a matrix

1: Find the minor of 2 and 4 from the matrix $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

Minor of $2 = |8| = 8$

Minor of $4 = |6| = 6$

2: Find the minor of -2 and -5 from the matrix $\begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & -3 \\ -2 & -5 \end{bmatrix}$

Minor of $-2 = |-3| = -3$

Minor of $-5 = |1| = 1$

3: Find the minor of 1 from the matrix $\begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 7 \\ 8 & 9 & 5 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 7 \\ 8 & 9 & 5 \end{bmatrix}$

Minor of 1=$\begin{vmatrix} 4 & 7 \\ 9 & 5 \end{vmatrix} = (4 \times 5) - (7 \times 9) = 20 - 63 = -43$

4: Find the minor of 5 and -3 from the matrix $\begin{bmatrix} 2 & 5 & 6 \\ -1 & 4 & -3 \\ 0 & 9 & 7 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & 5 & 6 \\ -1 & 4 & -3 \\ 0 & 9 & 7 \end{bmatrix}$

Minor of 5 = $\begin{vmatrix} -1 & -3 \\ 0 & 7 \end{vmatrix} = (-1 \times 7) - (-3 \times 0) = -7 - 0 = -7$

Minor of -3 = $\begin{vmatrix} 2 & 5 \\ 0 & 9 \end{vmatrix} = (2 \times 9) - (5 \times 0) = 18 - 0 = 18$

Cofactor of an element: Cofactor of an element is minor of the element with + or – sign given to it. If the element is in the i^{th} row and j^{th} column, the sign to be allocated is $(-1)^{i+j}$.

Steps to find Cofactor of an element

For each element of the matrix

Step (1): Allocate the + or – sign by using $(-1)^{i+j}$ or $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ or $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Step (2): Delete the elements on the current row and column

Step (3): Calculate the determinant of remaining elements

Example 1: Consider a 2×2 matrix A = $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

To find Cofactor of a_1

Step (1): Allocate + or – sign by using $(-1)^{i+j}$ or

Step (2): $+ \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

Step (3): $+|b_2| = b_2$

Follow the same method to find cofactors of remaining elements

C0factor of $\mathbf{a_1}$(C of $\mathbf{a_1}$) $= +|b_2| = A_1$

C of $\mathbf{a_2}$ $= -|b_1| = A_2$

C of $\mathbf{b_1}$ $= -|a_2| = B_1$

C of $\mathbf{b_2}$ $= +|a_1| = B_2$

$\therefore$ Cofcator matrix of $A = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}$

Example 2: Consider a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

To find Cofactor of a_1

Step (1): Allocate the + or – sign by using $(-1)^{i+j}$ or

Step (2): $+ \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Follow the same method to find cofactors of remaining

Step (3): $+ \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = +(b_2c_3 - b_3c_2)$

Cofactor of $\mathbf{a_1}$(C of $\mathbf{a_1}$) $= + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = +(b_2c_3 - c_2b_3) = b_2c_3 - b_3c_2 = A_1$

C of $\mathbf{a_2}$ $= - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -(b_1c_3 - b_3c_1) = -b_1c_3 + b_3c_1 = A_2$

C of $\mathbf{a_3}$ $= + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = +(b_1c_2 - b_2c_1) = b_1c_2 - b_2c_1 = A_3$

C of $\mathbf{b_1}$ $= - \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -(a_2c_3 - a_3c_2) = -a_2c_3 + a_3c_2 = B_1$

C of $\mathbf{b_2}$ $= + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = +(a_1c_3 - a_3c_1) = a_1c_3 - a_3c_1 = B_2$

C of $\mathbf{b_3}$ $= - \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -(a_1c_2 - a_2c_1) = -a_1c_2 + a_2c_1 = B_3$

C of $\mathbf{c_1}$ $= + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = +(a_2b_3 - a_3b_2) = a_2b_3 - a_3b_2 = C_1$

C of c_2 $= -\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = -(a_1c_3 - a_3c_1) = -a_1c_3 + a_3c_1 = C_2$

C of c_3 $= +\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = +(a_1b_2 - a_2b_1) = a_1b_2 - a_2b_1 = C_3$

$\therefore$ Cofcator matrix of A $= \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

Problems on finding cofactor of an element of a matrix

1: Find the cofactor of 3 and 5 from the matrix $\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$

Note: use these signs $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ **to find the cofactor of an element**

Cofactor of 3 $= +|9| = 9$

Cofactor of 5 $= -|7| = -7$

2: Find the cofactor matrix of $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

Cofactor of 1(**C of 1**) $= +|5| = 5$

C of 3 $= -|2| = -2$

C of 2 $= -|3| = -3$

C of 5 $= +|1| = 1$

$\therefore$ Cofcator matrix of A $= \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$

3: Find the cofactor of 5 and 6 from the matrix $\begin{bmatrix} 3 & 2 & 5 \\ 7 & 8 & 2 \\ 1 & 6 & 4 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 7 & 8 & 2 \\ 1 & 6 & 4 \end{bmatrix}$

Cofactor of 5 $= +\begin{vmatrix} 7 & 8 \\ 1 & 6 \end{vmatrix} = +(42 - 8) = +(34) = 34$

Cofactor of 6 $= -\begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix} = -(6 - 35) = -(-29) = 29$

4: Find the cofactor matrix of $\begin{bmatrix} 5 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 5 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix}$

Note: use these signs $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ **to find the cofactor of an element**

Cofactor of 5 (C of 5) $= +\begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = +(8-6) = +(2) = 2$

C of 1 $= -\begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} = -(16-12) = -(4) = -4$

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C of 3 $= +\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = +(4-4) = 0$

C of 4 $= -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -(4-3) = -(1) = 1$

C of 2 $= +\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} = +(20-6) = +(14) = 14$

C of 6 $= -\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} = -(5-2) = -(3) = -3$

C of 2 $= +\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = +(6-6) = 0$

C of 1 $= -\begin{vmatrix} 5 & 3 \\ 4 & 6 \end{vmatrix} = -(30-12) = -(18) = -18$

C of 4 $= +\begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = +(10-4) = +(6) = 6$

$\therefore$ Cofactor matrix of A $= \begin{bmatrix} 2 & -4 & 0 \\ 1 & 14 & -3 \\ 0 & -18 & 6 \end{bmatrix}$

Model questions(4marks)

1. Find the cofactor matrix of $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 4 & 2 & 3 \end{bmatrix}$
2. Find the cofactor matrix of $\begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 9 \\ 1 & 5 & 4 \end{bmatrix}$
3. If $A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 0 & 7 \\ 2 & -3 & -2 \end{bmatrix}$ Find the cofactor matrix of A

Adjoint of a matrix: Let A be square matrix of order n. The adjoint of a matrix A is the transpose of the cofactor matrix of A. It is denoted by **adj A.**

i.e. **adj A = [cofactor matrix of A]^T**

Steps to find adjoint of a matrix

Step (1): Find the cofactors of every element in the given matrix.

Step (2): Write down the cofactor matrix.

Step (3): Find the transpose of the cofactor matrix.

Example 1: Consider a 2×2 matrix $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

Cofactor matrix of A $= \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}$

NOTE: Where A_1, A_2, B_1 and B_2 are the cofactors of a_1, a_2, b_1 and b_2 respectively.

We know that, Adjoint of A(adjA)= [cofactor matrix of A] ^{T}

$$= \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}^T$$

Therefore, Adjoint of A$= \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$

Example 2: Consider a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Cofactor matrix of A $= \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

NOTE: Where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3 are the cofactors of $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$ and c_3 respectively.

We know that, **Adjoint of A(adjA)= [cofactor matrix of A]** ^{T}

$$= \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}^T$$

Therefore, Adjoint of A $= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

NOTE: For any square matrix of A, $\mathbf{A(adjA) = (adjA)A = |A|I}$

Problems on finding adjoint of a matrix

1: If $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$ find Adjoint of matrix A.

Solution: Given $A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$ $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

Cofactor of 4 (C of 4) $= +|1| = 1$

C of 2 $= -|5| = -5$

C of 5 $= -|2| = -2$

C of 1 $= +|4| = 4$

$\therefore$ Cofactor matrix of A $= \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$

We know that, adjA= [cofactor matrix of A] ^{T}

$$\text{adj}A = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$$

NOTE: Adjoint of 2×2 matrix can be found out by interchange the principal diagonal elements (PDE) and interchange the signs of remaining elements.

For example, If $A = \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix}$ PDE , $\text{adj}A = \begin{bmatrix} 3 & -7 \\ -9 & 2 \end{bmatrix}$

2: Find the adjoint of the matrix $A = \begin{bmatrix} 6 & 5 \\ -1 & 8 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 6 & 5 \\ -1 & 8 \end{bmatrix}$ $\qquad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$\textbf{C of 6} = +|8| = 8$

$\textbf{C of 5} = -|-1| = +1$

$\textbf{C of} -\textbf{1} = -|5| = -5$

$\textbf{C of 8} = +|6| = 6$

Cofactor matrix of A $= \begin{bmatrix} 8 & 1 \\ -5 & 6 \end{bmatrix}$

We know that, adj A= [cofactor matrix of A]^T

$$\text{adj}A = \begin{bmatrix} 8 & 1 \\ -5 & 6 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} 8 & -5 \\ 1 & 6 \end{bmatrix}$$

3: If $A = \begin{bmatrix} 3 & 1 & 5 \\ 6 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ find adjA.

Solution: Given $A = \begin{bmatrix} 3 & 1 & 5 \\ 6 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$\textbf{C of 3} = +\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = +(1-4) = +(-3) = -3$

$\textbf{C of 1} = -\begin{vmatrix} 6 & 2 \\ 4 & 1 \end{vmatrix} = -(6-8) = -(-2) = 2$

$\textbf{C of 5} = +\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = +(12-4) = +(8) = 8$

$\textbf{C of 6} = -\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -(1-10) = -(-9) = 9$

$\textbf{C of 1} = +\begin{vmatrix} 3 & 5 \\ 4 & 1 \end{vmatrix} = +(3-20) = +(-17) = -17$

$\textbf{C of 2} = -\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = -(6-4) = -(2) = -2$

$\textbf{C of 4} = +\begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} = +(2-5) = +(-3) = -3$

$\mathbf{C\ of\ 2} = -\begin{vmatrix} 3 & 5 \\ 6 & 2 \end{vmatrix} = -(6-30) = -(-24) = 24$

$\mathbf{C\ of\ 1} = +\begin{vmatrix} 3 & 1 \\ 6 & 1 \end{vmatrix} = +(3-6) = +(-3) = -3$

$\therefore$ Cofactor matrix of A $= \begin{bmatrix} -3 & 2 & 8 \\ 9 & -17 & -2 \\ -3 & 24 & -3 \end{bmatrix}$

We know that, adj A= [cofactor matrix of A] ^{T}

$$\text{adj}A = \begin{bmatrix} -3 & 2 & 8 \\ 9 & -17 & -2 \\ -3 & 24 & -3 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} -3 & 9 & -3 \\ 2 & -17 & 24 \\ 8 & -2 & -3 \end{bmatrix}$$

4: Find the adjoint of $\begin{bmatrix} 3 & -1 & 2 \\ 2 & -3 & 1 \\ 0 & 4 & 2 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & -3 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$\mathbf{Cof\ 3} = +\begin{vmatrix} -3 & 1 \\ 4 & 2 \end{vmatrix} = +(-6-4) = +(-10) = -10$

$\mathbf{C\ of\ -1} = -\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = -(4-0) = -(4) = -4$

$\mathbf{C\ of\ 2} = +\begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix} = +(8-0) = +(8) = 8$

$\mathbf{C\ of\ 2} = -\begin{vmatrix} -1 & 2 \\ 4 & 2 \end{vmatrix} = -(-2-8) = -(-10) = 10$

$\mathbf{C\ of\ -3} = +\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = +(6-0) = +(6) = 6$

$\mathbf{C\ of\ 1} = -\begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} = -(12-0) = -(12) = -12$

$\mathbf{C\ of\ 0} = +\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = +(-1+6) = +(5) = 5$

$\mathbf{C\ of\ 4} = -\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -(3-4) = -(-1) = 1$

$\mathbf{C\ of\ 2} = +\begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix} = +(-9+2) = +(-7) = -7$

$\therefore$ Cofactor matrix of A $= \begin{bmatrix} -10 & -4 & 8 \\ 10 & 6 & -12 \\ 5 & 1 & -7 \end{bmatrix}$

We know that, adj A= [cofactor matrix of A] ^{T}

$$\text{adj}A = \begin{bmatrix} -10 & -4 & 8 \\ 10 & 6 & -12 \\ 5 & 1 & -7 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} -10 & 10 & 5 \\ -4 & 6 & 1 \\ 8 & -12 & -7 \end{bmatrix}$$

Model questions (4marks)

1. Find the adjoint of the matrix $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$
2. If $A = \begin{bmatrix}2 & 1\\-3 & 3\end{bmatrix}$ find adjA
3. If $A = \begin{bmatrix}5 & 1\\6 & 8\end{bmatrix}$ find Adjoint of A

Model questions (5marks)

1. Find the adjoint of the matrix $\begin{bmatrix}1 & 2 & 3\\1 & 3 & 3\\2 & 4 & 3\end{bmatrix}$
2. If $A = \begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\-1 & 1 & -3\end{bmatrix}$ find Adjoint of A
3. If $A = \begin{bmatrix}3 & 0 & -2\\0 & 1 & 4\\0 & 2 & 3\end{bmatrix}$ find adjA

Singular matrix: A square matrix 'A' is said to be singular if and only if $|A| = 0$

Example: $A = \begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}$ is a singular matrix because $|A| = 0$

Non-singular matrix: A square matrix 'A' is said to be non-singular if and only if $|A| \neq 0$

Example: $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ is a non-singular matrix because $|A| \neq 0$

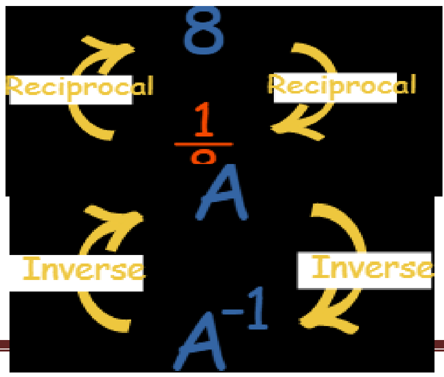
Inverse of a matrix: If a matrix 'A' is non-singular ($|A| \neq 0$), the inverse of matrix A can be defined.

i.e. $\mathbf{A^{-1} = \frac{1}{|A|}\, adjA}$

What is the Inverse of a Matrix?

This is the reciprocal of a number:

The **Inverse of a Matrix** is the same idea but we write it A^{-1}



Why not $\frac{1}{A}$? Because we don't divide by a matrix!

Why Do We Need an Inverse?

Because with matrices we **don't divide**! Seriously, there is no concept of dividing by a matrix.

But we can **multiply by an inverse**, which achieves the same thing.

Steps to find inverse of a matrix

Step (1): Find the determinant of a given matrix ($|A| \neq 0$).

Step (2): Find the adjoint of a given matrix.

Step (3): substitute $|A|$ value and adjoint of A in the formula $A^{-1} = \frac{1}{|A|}\,\mathrm{adj}A$

NOTE 1: If A and B are two non-singular matrices of same order then $(AB)^{-1} = B^{-1}A^{-1}$

NOTE 2: $\mathbf{AA^{-1} = A^{-1}A = I}$

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Problems on finding inverse of a matrix

1: Find inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$

Consider $|A| = \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = (8 - 6) = 2 \neq 0$

Therefore A^{-1} exists.

To find **adjA** interchange the principal diagonal elements and interchange the signs of remaining elements.

From the matrix A, adjA=$\begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}$

We know that, $\mathbf{A^{-1} = \frac{1}{|A|}\,adjA}$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}$$

2: Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 4 & -1 & -2 \end{bmatrix}$

Consider $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 4 & -1 & -2 \end{vmatrix} = 1(-2+1) + 1(-4-4) + 2(-2-4)$ = -21≠ 0

Therefore A^{-1} exists.

To find adjoint of matrix A

$\mathbf{C\ of\ 1} = +\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = +(-2+1) = +(-1) = -1$

$\mathbf{C\ of\ -1} = -\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -(-4-4) = -(-8) = 8$

$\mathbf{C\ of\ 2} = +\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = +(-2-4) = +(-6) = -6$

$\mathbf{C\ of\ 2} = -\begin{vmatrix} -1 & 2 \\ -1 & -2 \end{vmatrix} = -(2+2) = -(4) = -4$

$\mathbf{C\ of\ 1} = +\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = +(-2-8) = +(-10) = -10$

$\mathbf{C\ of\ 1} = -\begin{vmatrix} 1 & -1 \\ 4 & -1 \end{vmatrix} = -(-1+4) = -(3) = -3$

$\mathbf{C\ of\ 4} = +\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = +(-1-2) = +(-3) = -3$

$\mathbf{C\ of\ -1} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -(1-4) = -(-3) = 3$

$\mathbf{C\ of\ -2} = +\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = +(1+2) = +(3) = 3$

We know that, **adjA= [cofactor matrix of A]** $^{\mathbf{T}}$

$$\text{adjA} = \begin{bmatrix} -1 & 8 & -6 \\ -4 & -10 & -3 \\ -3 & 3 & 3 \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} -1 & -4 & -3 \\ 8 & -10 & 3 \\ -6 & -3 & 3 \end{bmatrix}$$

We know that, $\mathbf{A^{-1}} = \frac{\mathbf{1}}{|\mathbf{A}|}\ \mathbf{adjA}$

$$A^{-1} = -\frac{1}{21}\begin{bmatrix} -1 & -4 & -3 \\ 8 & -10 & 3 \\ -6 & -3 & 3 \end{bmatrix}$$

3: If $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 2 \\ 5 & 1 & -2 \end{bmatrix}$ find $\mathbf{A^{-1}}$

Solution: Given $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 2 \\ 5 & 1 & -2 \end{bmatrix}$

Consider $|A| = \begin{vmatrix} 1 & 2 & 4 \\ 3 & -1 & 2 \\ 5 & 1 & -2 \end{vmatrix} = 1(2-2) - 2(-6-10) + 4(3+5) = 64 \neq 0$

Therefore A^{-1} exists.

To find adjoint of matrix A

$\mathbf{C\ of\ 1} = +\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} = +(2-2) = +(0) = 0$

$\mathbf{C\ of\ 2} = -\begin{vmatrix} 3 & 2 \\ 5 & -2 \end{vmatrix} = -(-6-10) = -(-16) = 16$

$\mathbf{C\ of\ 4} = +\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} = +(3+5) = +(8) = 8$

$\mathbf{C\ of\ 3} = -\begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -(-4-4) = -(-8) = 8$

$\mathbf{C\ of} - \mathbf{1} = +\begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} = +(-2-20) = +(-22) = -22$

$\mathbf{C\ of\ 2} = -\begin{vmatrix} 1 & 2 \\ 5 & 1 \end{vmatrix} = -(1-10) = -(-9) = 10$

$\mathbf{C\ of\ 5} = +\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = +(4+4) = +(8) = 8$

$\mathbf{C\ of\ 1} = -\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -(2-12) = -(-10) = 10$

$\mathbf{C\ of} - \mathbf{2} = +\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = +(-1-6) = +(-7) = -7$

We know that, **adjA= [cofactor matrix of A]** $^{\mathbf{T}}$

$$\text{adjA} = \begin{bmatrix} 0 & 16 & 8 \\ 8 & -22 & 10 \\ 8 & 10 & -7 \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} 0 & 8 & 8 \\ 16 & -22 & 10 \\ 8 & 10 & -7 \end{bmatrix}$$

We know that, $\mathbf{A^{-1}} = \frac{\mathbf{1}}{|\mathbf{A}|}\ \mathbf{adjA}$

$$A^{-1} = \frac{1}{64}\begin{bmatrix} 0 & 8 & 8 \\ 16 & -22 & 10 \\ 8 & 10 & -7 \end{bmatrix}$$

Model questions (5marks)

1. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$
2. If $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ find the inverse of A

Model questions (6marks)

1. Find the inverse of the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ find the inverse of A

3. If $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 3 & -1 & 5 \end{bmatrix}$ find A^{-1}

Characteristic equation of matrix: Let A be square matrix of order n then $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = \mathbf{0}$ is called characteristic equation of matrix A. Where I is identity matrix of order n and λ is constant.

Characteristic roots of matrix: Let A be square matrix of order n then the roots of the characteristic equation $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = \mathbf{0}$ are called characteristic roots or eigen values.

NOTE: The sum of the eigen values is equal to the sum of the principal diagonal elements of the matrix.

Problems on finding characteristic equation and eigen values of a

1: Find the characteristic equation of $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

C.E is given by $|A - \lambda I| = 0$

$\left|\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$ (Where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity matrix)

$\begin{vmatrix} 2-\lambda & -1 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda) + 3 = 0$

$4 - 2\lambda - 2\lambda + \lambda^2 + 3 = 0$

$\lambda^2 - 4\lambda + 7 = 0$ is required characteristic equation.

2: Find the characteristic equation of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

C.E is given by $|A - \lambda I| = 0$

$\left|\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - 6 = 0$

$4 - \lambda - 4\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 5\lambda - 2 = 0$ is required characteristic equation.

3: Find the characteristic equation and eigen values of $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

C.E is given by $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$(3-\lambda)(5-\lambda) - 8 = 0$

$15 - 3\lambda - 5\lambda + \lambda^2 - 8 = 0$

$\lambda^2 - 8\lambda + 7 = 0$

$\lambda^2 - 7\lambda - 1\lambda + 7 = 0$

$\lambda(\lambda - 7) - 1(\lambda - 7) = 0$

$(\lambda - 7)(\lambda - 1) = 0$

$\lambda - 7 = 0$ or $\lambda - 1 = 0$

$\lambda = 7$ or $\lambda = 1$

$\lambda = 7,1$ are the eigen values of given matrix.

4: If $A = \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}$ find the eigen values

Solution: Given $A = \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix}$

C.E is given by $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 3 & -1 \\ 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 0 & -2-\lambda \end{vmatrix} = 0$$

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$(3-\lambda)(-2-\lambda) - 0 = 0$

$-6 - 3\lambda + 2\lambda + \lambda^2 - 0 = 0$

$\lambda^2 - \lambda - 6 = 0$

$\lambda^2 - 3\lambda + 2\lambda - 6 = 0$

$\lambda(\lambda - 3) + 2(\lambda - 3) = 0$

$(\lambda - 3)(\lambda + 2) = 0$

$\lambda - 3 = 0$ or $\lambda + 2 = 0$

$\lambda = 3$ or $\lambda = -2$

$\lambda = 3,-2$ are the eigen values of given matrix.

5: Find the characteristic equation and its roots of $A = \begin{bmatrix} 1 & -1 \\ -6 & -2 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & -1 \\ -6 & -2 \end{bmatrix}$

C.E is given by $|A - \lambda I| = 0$

$\left| \begin{bmatrix} 1 & -1 \\ -6 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$

$\begin{vmatrix} 1-\lambda & -1 \\ -6 & -2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(-2-\lambda) - 6 = 0$

$-2 - 1\lambda + 2\lambda + \lambda^2 - 6 = 0$

$\lambda^2 + 1\lambda - 8 = 0$

The above equation is the form of quadratic equation $ax^2 + bx + c = 0$

By using formula $X = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

We get $\lambda = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-8)}}{2(1)}$

$\lambda = \frac{-1 \pm \sqrt{1+32}}{2}$

$\lambda = \frac{-1 \pm \sqrt{33)}}{2}$

$\lambda = \frac{-1+\sqrt{33)}}{2}, \frac{-1-\sqrt{33)}}{2}$ are the characteristic roots of given matrix.

Model questions (6marks)

1. Find the characteristic equation and roots for the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$
2. Find the characteristic equation and eigen values for the matrix $\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$
3. Find the characteristic equation and eigen values for the matrix $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$
4. Find the eigen values for the matrix $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$

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